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COMMENT

On conformally covariant spin-$\frac{3}{2}$ and spin-2 equations

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Abstract. It is demonstrated that the conformally covariant spin-$\frac{3}{2}$ and spin-2 equations are reducible.

In some recent papers (Barut and Xu 1982, Drew and Gegenberg 1980, Drew 1983, Xu 1982) conformally covariant spin-$\frac{3}{2}$ and spin-2 equations were proposed and studied. It appears that from the viewpoint of wave equations covariant under the Lorentz group the conformally covariant equations are reducible. In this comment we clarify the general structure of the given equations.

Barut and Xu (1982) interpreted the difference between the conformally covariant equation and the usual equation as a source term to be placed on the right-hand side of the usual equation. Here we demonstrate that such a source term contains only lower-spin components.

The conformally covariant spin-$\frac{3}{2}$ equation is (Barut and Xu 1982)

$$\eth\psi^{\varkappa}-\tfrac{1}{2}(\partial^{\varkappa}\gamma_{\lambda}+\gamma^{\varkappa}\partial_{\lambda})\psi^{\lambda}+\gamma^{\varkappa}\eth\gamma_{\lambda}\psi^{\lambda}=0. \tag{1}$$

In order to demonstrate the general structure of a given equation it is useful to write it in a matrix form using the formalism of covariant spin-projection operators (Loide 1984, 1985):

$$\sqrt{\Box}\begin{vmatrix}\beta_{11}^{3/2}+\frac{1}{2}\beta_{11}^{1/2} & a\beta_{12}^{1/2}\\ b\beta_{21}^{1/2} & c\beta_{22}^{1/2}\end{vmatrix}\begin{vmatrix}\psi_1\\ \psi_2\end{vmatrix}=0. \tag{2}$$

Here we have decomposed $\psi^{\varkappa}$ into a direct sum $\psi_1\oplus\psi_2$, where ψ_1 transforms according to the representation $1=(1,\frac{1}{2})\oplus(\frac{1}{2},1)$ and ψ_2 transforms according to the representation $2=(\frac{1}{2},0)\oplus(0,\frac{1}{2})$.

Using the results of Loide (1984) we obtain that the conformally covariant equation (1) corresponds to $a=b=0$ and $c=\frac{5}{2}$. We therefore have two independent equations for fields ψ_1 and ψ_2. The field ψ_1 describes spin-$\frac{3}{2}$ and spin-$\frac{1}{2}$ states and the field ψ_2 the spin-$\frac{1}{2}$ state. The conformally covariant equation is not gauge invariant. For gauge invariance it is necessary that a, b and c are non-zero and $ab=c/2$ (Loide and Polt 1985).

The massless Rarita-Schwinger equation corresponds to the choice $a=b=-\frac{1}{2}\sqrt{3}$ and $c=\frac{3}{2}$. Barut and Xu (1982) interpreted the difference between the Rarita-Schwinger equation and equation (1),

$$\begin{aligned}\eta^{\varkappa} &= -\sqrt{\Box}[(\tfrac{1}{2}\sqrt{3}\beta_{12}^{1/2}+\beta_{22}^{1/2})^{\varkappa}{}_{\lambda}\psi_2^{\lambda}+(\tfrac{1}{2}\sqrt{3}\beta_{21}^{1/2})^{\varkappa}{}_{\lambda}\psi_1^{\lambda}]\\ &= -\tfrac{1}{2}(\partial^{\varkappa}\gamma_{\lambda}\psi^{\lambda}+\gamma^{\varkappa}\partial_{\lambda}\psi^{\lambda}),\end{aligned} \tag{3}$$

as a source term in the Rarita–Schwinger equation. Contrary to the assertion of Barut and Xu (1982) the above given term includes only spin-$\frac{1}{2}$ components.

The source constraint $\partial_\varkappa \eta^\varkappa = 0$ implies additional restrictions on ψ. In terms of projection operators it is possible to verify that the spin-$\frac{1}{2}$ components in representations ψ_1 and ψ_2 are not independent but connected. Therefore the source in the Rarita–Schwinger equation which has the structure (3) is a spin-$\frac{1}{2}$ source with one spin-$\frac{1}{2}$ component.

The conformally covariant spin-2 equation can be analysed similarly to the previous case. The conformally covariant spin-2 equation is (Barut and Xu 1982, Xu 1982, Drew 1983)

$$\Box h^{\varkappa\lambda} - \tfrac{2}{3}(\partial^\varkappa \partial_\rho h^{\rho\lambda} + \partial^\lambda \partial_\rho h^{\rho\varkappa}) + \tfrac{1}{3}\partial^\varkappa \partial^\lambda h^\rho{}_\rho + \tfrac{1}{3}\eta^{\varkappa\lambda}(\partial_\rho \partial_\sigma h^{\rho\sigma} - \Box h^\rho{}_\rho) = 0. \tag{4}$$

Using the spin-projection operators P^s_{ij} (Loide 1985), equation (4) is written in the following matrix form:

$$\Box \begin{vmatrix} P^2_{11} + \alpha P^1_{11} + \beta P^0_{11} & bP^0_{12} \\ aP^0_{21} & cP^0_{22} \end{vmatrix} \begin{vmatrix} h_1 \\ h_2 \end{vmatrix} = 0. \tag{5}$$

We have decomposed $h^{\varkappa\lambda}$ into a direct sum $h_1 \oplus h_2$, where h_1 transforms according to the representation $1 = (1, 1)$ and h_2 transforms according to $2 = (0, 0)$, i.e. $(h_1)^{\varkappa\lambda} = h^{\varkappa\lambda} - \frac{1}{4}\eta^{\varkappa\lambda} h^\rho{}_\rho$, $(h_2)^{\varkappa\lambda} = \frac{1}{4}\eta^{\varkappa\lambda} h^\rho{}_\rho$.

The conformally covariant equation (4) corresponds to $\alpha = \frac{1}{3}$, $\beta = 0$, $a = b = c = 0$. We therefore have the equation for field h_1; field h_2 is absent. Due to the presence of the spin-1 projection operator P^1_{11}, spin-1 components are also described by equation (4).

The massless Pauli–Fierz equation corresponds to the choice $\alpha = 0$, $\beta = -\frac{1}{2}$, $a = b = -\frac{1}{2}\sqrt{3}$, $c = -\frac{3}{2}$. The main difference between the Pauli–Fierz equation and the conformally covariant equation is in spin-1 components. In the Pauli–Fierz equation the spin-projection operator P^1_{11} is absent. Due to the presence of the spin-projection operator P^0_{11} the equation is gauge invariant since a, b and c satisfy $ab = -\frac{1}{2}c$.

If we consider the difference between these two equations as a source term, we have

$$\begin{aligned} T^{\varkappa\lambda} &= -\Box[(\tfrac{1}{3}P^1_{11} + \tfrac{1}{2}P^0_{11} + \tfrac{1}{2}\sqrt{3}P^0_{21})^{\varkappa\lambda}{}_{\rho\sigma} h_1^{\rho\sigma} + (\tfrac{1}{2}\sqrt{3}P^0_{12} + \tfrac{3}{2}P^0_{22})^{\varkappa\lambda}{}_{\rho\sigma} h_2^{\rho\sigma}] \\ &= -\tfrac{1}{3}(\partial^\varkappa \partial_\rho h^{\rho\lambda} + \partial^\lambda \partial_\rho h^{\rho\varkappa}) + \tfrac{2}{3}\partial^\varkappa \partial^\lambda h^\rho{}_\rho + \tfrac{2}{3}\eta^{\varkappa\lambda}(\partial_\rho \partial_\sigma h^{\rho\sigma} - \Box h^\rho{}_\rho). \end{aligned} \tag{6}$$

This source term includes spin-1 and spin-0 components. The source constraint $\partial_\varkappa T^{\varkappa\lambda} = 0$ demands that the spin-1 component must be absent. Therefore the source in the Pauli–Fierz equation having the structure (6) must be a spin-0 source.

The conformally covariant spin-2 equation given by Drew and Gegenberg (1980) and Drew (1983) is the following:

$$\Box h^{\varkappa\lambda} - \tfrac{2}{3}(\partial^\varkappa \partial_\rho h^{\rho\lambda} + \partial^\lambda \partial_\rho h^{\rho\varkappa}) + \tfrac{1}{3}\partial^\varkappa \partial^\lambda h^\rho{}_\rho + \tfrac{1}{3}\eta^{\varkappa\lambda}\partial_\rho \partial_\sigma h^{\rho\sigma} = 0. \tag{7}$$

If we write the last equation as (5), we have $\alpha = \frac{1}{3}$, $\beta = 0$, $a = b = 0$, $c = \frac{4}{3}$. Now we have two independent equations for fields h_1 and h_2.

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